ON THE TRIANGLE INEQUALITY FOR THE STANDARD 2-NORM

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Abstract. We shall show here that the triangle inequality for the standard 2-norm is equivalent to a generalized Cauchy-Schwarz inequality, as is that for norms to the Cauchy-Schwarz inequality. Besides its direct proof, we also present two alternative proofs through its equivalent inequality.

1. Introduction. Let \( X \) be a real-vector space, equipped with an inner-product \( \langle \cdot, \cdot \rangle \) together with its induced norm \( \|x\| = \langle x, x \rangle^{\frac{1}{2}} \). Define the function \( \|\cdot, \cdot\| : X \times X \to \mathbb{R} \) by
\[
\|x, y\| := \left\{ \|x\|^2 \|y\|^2 - \langle x, y \rangle^2 \right\}^{\frac{1}{2}},
\]
which is equal to twice the area of the triangle having vertices 0, \( x \), and \( y \) (or the area of the parallelogram spanned by the vectors \( x \) and \( y \)) in \( X \).

It can be shown that the above function defines a 2-norm on \( X \), which satisfies the following four properties:

(i) \( \|x, y\| = 0 \) iff \( x \) and \( y \) are linearly dependent;
(ii) \( \|x, y\| = \|y, x\| \);
(iii) \( \|x, ay\| = |a| \|x, y\|, \ a \in \mathbb{R} \);
(iv) \( \|x, y + z\| \leq \|x, y\| + \|x, z\| \).

The space \( X \), being equipped with the 2-norm, is thus a 2-normed space.

The concepts of 2-normed spaces (and 2-metric spaces) were initially introduced by Gahler \([G1], [G2], [G3]\) in 1960’s. A standard example of a 2-normed space is \( \mathbb{R}^2 \) equipped

Keywords: 2-normed spaces, inner-product spaces, generalized Cauchy-Schwarz inequality
with the following 2-norm
\[ \|x, y\| := \text{the area of the triangle having vertices } 0, \ x, \ \text{and } y. \]

The 2-norm above on \( X \) is a just generalization of this standard example. For recent results on 2-normed spaces, see for example [GM].

As it usually happens with norms, given a candidate for a 2-norm, the hardest part is to check the property (iv) — better known as the triangle inequality. In this note, we shall show that the triangle inequality for the 2-norm above is equivalent to a generalized Cauchy-Schwarz inequality, just as is that for norms to the Cauchy-Schwarz inequality. Besides its direct proof, we also present two alternative proofs through its equivalent inequality.

2. The Inequality. For our 2-norm above on \( X \), we have the following fact:

**FACT.** The triangle inequality is equivalent to

\[ \|x\|^2 \langle y, z \rangle^2 + \|y\|^2 \langle x, z \rangle^2 + \|z\|^2 \langle x, y \rangle^2 \leq \|x\|^2 \|y\|^2 \|z\|^2 + 2 \langle x, y \rangle \langle x, z \rangle \langle y, z \rangle. \]

**PROOF.** Observe that

\[
\|x, y + z\|^2 = \|x\|^2 \|y + z\|^2 - \langle x, y + z \rangle^2 \\
= \|x\|^2 (\|y\|^2 + 2 \langle y, z \rangle + \|z\|^2) - (\langle x, y \rangle^2 + 2 \langle x, y \rangle \langle x, z \rangle + \langle x, z \rangle^2) \\
= \|x, y\|^2 + 2 (\|x\|^2 \langle y, z \rangle - \langle x, y \rangle \langle x, z \rangle) + \|x, z\|^2.
\]

The triangle inequality is thus equivalent to

\[ \|x\|^2 \langle y, z \rangle - \langle x, y \rangle \langle x, z \rangle \leq \|x\| \|y\| \|x, z\|. \]

Replacing \( y + z \) by \( y - z \), we find that the triangle inequality is equivalent to

\[ \|x\|^2 \langle y, z \rangle - \langle x, y \rangle \langle x, z \rangle \leq \|x, y\| \|x, z\|. \]
Squaring both sides, we get
\[
\|x\|^4 \langle y, z \rangle^2 - 2\|x\|^2 \langle x, y \rangle \langle x, z \rangle \langle y, z \rangle + \langle x, y \rangle^2 \langle x, z \rangle^2 \\
\leq \|x\|^4 \|y\|^2 \|z\|^2 - \|x\|^2 (\|y\|^2 \langle x, z \rangle^2 + \|z\|^2 \langle y, z \rangle^2) + \langle x, y \rangle^2 \langle x, z \rangle^2.
\]
Canceling \(\langle x, y \rangle^2 \langle x, z \rangle^2\) and then dividing both sides by \(\|x\|^2\) (assuming that \(x \neq 0\)), we obtain the desired inequality.

Note. It is clear from the proof that the equality \(\|x\|^2 \langle y, z \rangle^2 + \|y\|^2 \langle x, z \rangle^2 + \|z\|^2 \langle y, z \rangle^2 = \|x\|^2 \|y\|^2 \|z\|^2 + 2 \langle x, y \rangle \langle x, z \rangle \langle y, z \rangle\) holds iff \(\|x, y + z\| = \|x, y\| + \|x, z\|\) or \(\|x, y - z\| = \|x, y\| + \|x, z\|\) holds.

Let us take a closer look at the inequality. First note that the equality holds when \(x\) or \(y\) or \(z\) equals 0. One may also observe that the equality holds when \(x = \pm y\) or \(x = \pm z\) or \(y = \pm z\). Further, if \(z \perp \text{span}\{x, y\}\) and \(z \neq 0\), then the inequality becomes
\[
\langle x, y \rangle^2 \leq \|x\|^2 \|y\|^2,
\]
which is the Cauchy-Schwarz inequality. Hence the inequality may be viewed as a generalized Cauchy-Schwarz inequality.

For \(X = \mathbb{R}\), the equality obviously holds. For \(X = \mathbb{R}^2\), the equality also holds. To see this, assume \(x, y, z \neq 0\). Dividing both sides by \(\|x\|^2 \|y\|^2 \|z\|^2\), we obtain
\[
\frac{\langle x, y \rangle^2}{\|x\|^2 \|y\|^2} + \frac{\langle x, z \rangle^2}{\|x\|^2 \|z\|^2} + \frac{\langle y, z \rangle^2}{\|y\|^2 \|z\|^2} \leq 1 + 2 \frac{\langle x, y \rangle \langle x, z \rangle \langle y, z \rangle}{\|x\| \|y\| \|x\| \|z\| \|y\| \|z\|}
\]
Next, assuming that \(\|x\| = \|y\| = \|z\| = 1\), the inequality becomes
\[
\langle x, y \rangle^2 + \langle x, z \rangle^2 + \langle y, z \rangle^2 \leq 1 + 2 \langle x, y \rangle \langle x, z \rangle \langle y, z \rangle.
\]
Now put \(\alpha = \angle(x, y)\), \(\beta = \angle(x, z)\), and \(\gamma = \angle(y, z)\). Then \(\alpha + \beta + \gamma = 2\pi\), and we have the equality
\[
\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 + 2 \cos \alpha \cos \beta \cos \gamma.
\]
Alternatively, assume \( x \neq \pm y \), so that \( \text{span}\{x, y\} = \mathbb{R}^2 \). Writing \( z = ax + by \) for some \( a, b \in \mathbb{R} \), one may check that both sides are equal to

\[
1 + 2ab\langle x, y \rangle + (1 + a^2 + b^2)\langle x, y \rangle^2.
\]

In general, it can be shown that the equality holds iff \( \text{span}\{x, y, z\} \) is at most two dimensional.

3. The Proof. We shall now prove the inequality. There are at least three ways to do it. First, if \( X \) is separable, then we can verify the triangle inequality for the 2-norm directly. Let \((e_i)\) be an orthonormal basis for \( X \) (indexed by a countable set). Then, by Parseval’s formula and polarization identity, we have

\[
\|x, y\| = \left\{ \left( \sum_i \langle x, e_i \rangle^2 \right) \left( \sum_j \langle y, e_j \rangle^2 \right) - \left( \sum_i \langle x, e_i \rangle \langle y, e_i \rangle \right)^2 \right\}^{\frac{1}{2}}
= \frac{1}{2} \sum_i \sum_j (\langle x, e_i \rangle \langle y, e_j \rangle - \langle x, e_j \rangle \langle y, e_i \rangle)^2 \right\}^{\frac{1}{2}}.
\]

The triangle inequality then follows easily.

Second, whether or not \( X \) is separable, we can always prove its equivalent inequality as follows. As argued earlier, under the assumption \( \|x\| = \|y\| = \|z\| = 1 \), we only need to show

\[
\langle x, y \rangle^2 + \langle x, z \rangle^2 + \langle y, z \rangle^2 \leq 1 + 2\langle x, y \rangle \langle x, z \rangle \langle y, z \rangle.
\]

Assuming \( x \neq \pm y \) and \( z \) is not perpendicular to \( \text{span}\{x, y\} \), we may write \( z = z_1 + z_2 \) where \( z_1 \in \text{span}\{x, y\} \), that is \( z_1 = ax + by \) for some \( a, b \in \mathbb{R} \), and \( z_2 \perp \text{span}\{x, y\} \). As in \( \mathbb{R}^2 \), we then have the equality

\[
\langle x, y \rangle^2 + \langle x, n_1 \rangle^2 + \langle y, n_1 \rangle^2 = 1 + 2\langle x, y \rangle \langle x, n_1 \rangle \langle y, n_1 \rangle
\]

where \( n_1 = z_1/\|z_1\| \). Multiplying both sides by \( \|z_1\|^2 \), we get

\[
\|z_1\|^2 \langle x, y \rangle^2 + \langle x, z \rangle^2 + \langle y, z \rangle^2 = \|z_1\|^2 + 2\langle x, y \rangle \langle x, z \rangle \langle y, z \rangle.
\]

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since \( \langle x, z_1 \rangle = \langle x, z \rangle \) and \( \langle y, z_1 \rangle = \langle y, z \rangle \). Hence

\[
\langle x, z \rangle^2 + \langle y, z \rangle^2 - 2\langle x, y \rangle \langle x, z \rangle \langle y, z \rangle = \|z_1\|^2(1 - \langle x, y \rangle^2) \leq 1 - \langle x, y \rangle^2,
\]

since \( \|z_1\| \leq \|z\| = 1 \) and \( 1 - \langle x, y \rangle^2 \geq 0 \); and the equality holds iff \( \|z_1\| = 1 \) (and consequently \( z_2 = 0 \)), that is iff \( z \in \text{span}\{x, y\} \).

Third, one may observe that our inequality is actually equivalent to

\[
\det(M) \geq 0
\]

where \( M \) is the Gram matrix

\[
M = \begin{pmatrix}
\langle x, x \rangle & \langle x, y \rangle & \langle x, z \rangle \\
\langle y, x \rangle & \langle y, y \rangle & \langle y, z \rangle \\
\langle z, x \rangle & \langle z, y \rangle & \langle z, z \rangle
\end{pmatrix}.
\]

Since \( M \) is positive semidefinite, the inequality follows immediately. It is also easy to see here that the equality holds iff \( \text{span}\{x, y, z\} \) is at most two dimensional (see [HJ, pp. 407-408]).

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