INNER PRODUCTS ON $n$-INNER PRODUCT SPACES

BY

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Abstract. In this note, we show that in any $n$-inner product space with $n \geq 2$ we can explicitly derive an inner product or, more generally, an $(n - k)$-inner product from the $n$-inner product, for each $k \in \{1, \ldots, n-1\}$. We also present some related results on $n$-normed spaces.

1. Introduction

Let $n$ be a nonnegative integer and $X$ be a real vector space of dimension $d \geq n$ ($d$ may be infinite). A real-valued function $\langle \cdot, \cdot, \ldots, \cdot \rangle$ on $X^{n+1}$ satisfying the following five properties:

(I1) $\langle x_1, x_1|x_2, \ldots, x_n \rangle \geq 0$; $\langle x_1, x_1|x_2, \ldots, x_n \rangle = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent;

(I2) $\langle x_1, x_1|x_2, \ldots, x_n \rangle = \langle x_{i_1}, x_{i_1}|x_{i_2}, \ldots, x_{i_n} \rangle$ for every permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$;

(I3) $\langle x, y|x_2, \ldots, x_n \rangle = \langle y, x|x_2, \ldots, x_n \rangle$;

(I4) $\langle \alpha x, y|x_2, \ldots, x_n \rangle = \alpha \langle x, y|x_2, \ldots, x_n \rangle$, $\alpha \in \mathbb{R}$;

(I5) $\langle x + x', y|x_2, \ldots, x_n \rangle = \langle x, y|x_2, \ldots, x_n \rangle + \langle x', y|x_2, \ldots, x_n \rangle$,

is called an $n$-inner product on $X$, and the pair $(X, \langle \cdot, \cdot, \ldots, \cdot \rangle)$ is called an $n$-inner product space.

For $n = 1$, the expression $\langle x, y|x_2, \ldots, x_n \rangle$ is to be understood as $\langle x, y \rangle$, which denotes nothing but an inner product on $X$. The concept of 2-inner product spaces was first introduced by Diminnie, Gähler and White [2, 3, 7] in 1970's.
while its generalization for \( n \geq 2 \) was developed by Misiak [12] in 1989. Note here that our definition of \( n \)-inner products is slightly simpler than, but equivalent to, that in [12].

On an \( n \)-inner product space \((X, \langle \cdot, \cdot |, \ldots, \cdot \rangle)\), one may observe that the following function

\[
\| x_1, x_2, \ldots, x_n \| := \langle x_1, x_1 | x_2, \ldots, x_n \rangle^{1/2},
\]

defines an \( n \)-norm, which enjoys the following four properties:

(N1) \( \| x_1, \ldots, x_n \| \geq 0; \| x_1, \ldots, x_n \| = 0 \) if and only if \( x_1, \ldots, x_n \) are linearly dependent;
(N2) \( \| x_1, \ldots, x_n \| \) is invariant under permutation;
(N3) \( \| \alpha x_1, x_2, \ldots, x_n \| = |\alpha| \| x_1, x_2, \ldots, x_n \|, \alpha \in \mathbb{R}; \)
(N4) \( \| x + y, x_2, \ldots, x_n \| \leq \| x, x_2, \ldots, x_n \| + \| y, x_2, \ldots, x_n \|. \)

Just as in an inner product space, we have the Cauchy-Schwarz inequality:

\[
|\langle x, y | x_2, \ldots, x_n \rangle| \leq \| x, x_2, \ldots, x_n \| \| y, x_2, \ldots, x_n \|,
\]

and the equality holds if and only if \( x, y, x_2, \ldots, x_n \) are linearly dependent (see [9]). Furthermore, we have the polarization identity:

\[
\| x + y, x_2, \ldots, x_n \|^2 - \| x - y, x_2, \ldots, x_n \|^2 = 4 \langle x, y | x_2, \ldots, x_n \rangle,
\]

and the parallelogram law:

\[
\| x + y, x_2, \ldots, x_n \|^2 + \| x - y, x_2, \ldots, x_n \|^2 = 2(\| x, x_2, \ldots, x_n \|^2 + \| y, x_2, \ldots, x_n \|^2).
\]

The latter gives a characterization of \( n \)-inner product spaces.

By the polarization identity and the property (I2), one may observe that

\[
\langle x, y | x_2, \ldots, x_n \rangle = \langle x, y | x_{i_2}, \ldots, x_{i_n} \rangle,
\]

for every permutation \( (i_2, \ldots, i_n) \) of \( (2, \ldots, n) \). Moreover, one can also show that

\[
\langle x, y | x_2, \ldots, x_n \rangle = 0,
\]

when \( x \) or \( y \) is a linear combination of \( x_2, \ldots, x_n \), or when \( x_2, \ldots, x_n \) are linearly dependent.
Now, for example, if \((X, \langle \cdot, \cdot \rangle)\) is an inner product space, then the following function
\[
\langle x, y| x_2, \ldots, x_n \rangle := \begin{vmatrix}
\langle x, y \rangle & \langle x, x_2 \rangle & \cdots & \langle x, x_n \rangle \\
\langle x_2, y \rangle & \langle x_2, x_2 \rangle & \cdots & \langle x_2, x_n \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle x_n, y \rangle & \langle x_n, x_2 \rangle & \cdots & \langle x_n, x_n \rangle 
\end{vmatrix},
\]
defines an \(n\)-inner product, called the standard (or simple) \(n\)-inner product on \(X\). Its induced \(n\)-norm \(\|x_1, x_2, \ldots, x_n\|\) represents the volume of the \(n\)-dimensional parallelepiped spanned by \(x_1, x_2, \ldots, x_n\).

Historically, the concept of \(n\)-norms were introduced earlier by Gähler in order to generalize the notion of length, area and volume in a real vector space (see \([4, 5, 6]\)). The objects studied here are \(n\)-dimensional parallelepipeds. The concept of \(n\)-inner products is thus useful when one talks about the angle between two \(n\)-dimensional parallelepipeds having the same \((n-1)\)-dimensional base.

In this note, we shall show that in any \(n\)-inner product space with \(n \geq 2\) we can derive an \((n-k)\)-inner product from the \(n\)-inner product for each \(k \in \{1, \ldots, n-1\}\). In particular, in any \(n\)-inner product space, we can derive an inner product from the \(n\)-inner product, so that one can talk about, for instance, the angle between two vectors, as one might like to.

In addition, we shall present some related results on \(n\)-normed spaces. See \([5]\) and \([11]\) for previous results on these spaces.

2. Main Results

To avoid confusion, we shall sometimes denote an \(n\)-inner product by \(\langle \cdot, \cdot | \cdot, \cdot \rangle_n\) and an \(n\)-norm by \(\| \cdot, \cdot \|_n\).

Let \((X, \langle \cdot, \cdot | \cdot, \cdot \rangle)_n\) be an \(n\)-inner product space with \(n \geq 2\). Fix a linearly independent set \(\{a_1, \ldots, a_n\}\) in \(X\). With respect to \(\{a_1, \ldots, a_n\}\), define for each \(k \in \{1, \ldots, n-1\}\) the function \(\langle \cdot, \cdot | \cdot, \cdot \rangle_{n-k}\) on \(X^{n-k+1}\) by
\[
\langle x, y|x_2, \ldots, x_{n-k} \rangle := \sum_{\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}} \langle x, y|x_2, \ldots, x_{n-k}, a_{i_1}, \ldots, a_{i_k} \rangle.
\]
Then we have the following fact:
Fact 2.1. For every \( k \in \{1, \ldots, n-1\} \), the function \( \langle \cdot, \cdot, \ldots, \cdot \rangle_{n-k} \) defines an \((n-k)\)-inner product on \( X \). In particular, when \( k = n - 1 \),

\[
\langle x, y \rangle := \sum_{\{i_2, \ldots, i_n\} \subseteq \{1, \ldots, n\}} \langle x, y | a_{i_2}, \ldots, a_{i_n} \rangle,
\]

(1)
defines an inner product on \( X \).

Proof. It is not hard to see that the function \( \langle \cdot, \cdot, \ldots, \cdot \rangle_{n-k} \) satisfies the five properties (I1)–(I5) of an \((n-k)\)-inner product, except perhaps to establish the second part of (I1). To verify this property, suppose that \( x_1, \ldots, x_{n-k} \) are linearly dependent. Then \( \langle x_1, x_1 | x_2, \ldots, x_{n-k}, a_{i_1}, \ldots, a_{i_k} \rangle = 0 \) for every \( \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\} \), and hence \( \langle x_1, x_1 | x_2, \ldots, x_{n-k} \rangle = 0 \). Conversely, suppose that \( \langle x_1, x_1 | x_2, \ldots, x_{n-k} \rangle = 0 \). Then \( \langle x_1, x_1 | x_2, \ldots, x_{n-k}, a_{i_1}, \ldots, a_{i_k} \rangle = 0 \) for every \( \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\} \). Hence \( x_1, \ldots, x_{n-k}, a_{i_1}, \ldots, a_{i_k} \) are linearly dependent for every \( \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\} \). By elementary linear algebra, this can only happen if \( x_1, \ldots, x_{n-k} \) are linearly dependent (or if \( x_1 = 0 \) when \( k = n - 1 \)).

Corollary 2.2. Any \( n \)-inner product space is an \((n-k)\)-inner product space for every \( k = 1, \ldots, n-1 \). In particular, an \( n \)-inner product space is an inner product space.

Corollary 2.3. Let \( \| \cdot, \ldots, \cdot \|_n \) be the induced \( n \)-norm on \( X \). Then, for each \( k \in \{1, \ldots, n-1\} \), the following function

\[
\|x_1, \ldots, x_{n-k}\| := \left( \sum_{\{i_2, \ldots, i_n\} \subseteq \{1, \ldots, n\}} \|x_1, \ldots, x_{n-k}, a_{i_2}, \ldots, a_{i_n}\|_n^2 \right)^{1/2},
\]
defines an \((n-k)\)-norm that corresponds to \( \langle \cdot, \cdot, \ldots, \cdot \rangle_{n-k} \) on \( X \). In particular,

\[
\|x\| := \left( \sum_{\{i_2, \ldots, i_n\} \subseteq \{1, \ldots, n\}} \|x, a_{i_2}, \ldots, a_{i_n}\|_n^2 \right)^{1/2},
\]
defines a norm that corresponds to the derived inner product \( \langle \cdot, \cdot \rangle \) on \( X \).

Note that by using a derived inner product, one can develop the notion of orthogonality and the Fourier series theory in an \( n \)-inner product space, just
like in an inner product space (see [3] and [13] for previous results in this direction). With respect to the derived inner product $\langle \cdot, \cdot \rangle$ defined by (1), one may observe that the set $\{a_1, \ldots, a_n\}$ is orthogonal and that $\|a_i\| = \|a_1, \ldots, a_n\|$ for every $i = 1, \ldots, n$ (see [8]). In particular, if $X$ is $n$-dimensional, then $\{a_1, \ldots, a_n\}$ forms an orthogonal basis for $X$ and each $x \in X$ can be written as $x = \|a_1, \ldots, a_n\|^{-2} \sum_{i=1}^{n} \langle x, a_i \rangle a_i$.

Unlike in [13], we can now have an orthogonal set of $m$ vectors with $1 \leq m < n$. In general, by using a derived inner product, we have a more relaxed condition for orthogonality than that in [3] or [13].

Furthermore, one may also use the derived inner products and their induced norm to study the convergence of sequences of vectors in an $n$-inner product space. See some recent results in [10].

2.1. Related results on $n$-normed spaces

Suppose now that $(X, \|\cdot, \ldots, \cdot\|_n)$ is an $n$-normed space and, as before, $\{a_1, \ldots, a_n\}$ is a linearly independent set in $X$. Then one may check that for each $k \in \{1, \ldots, n-1\}$

$$\|x_1, \ldots, x_{n-k}\| := \left( \sum_{\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}} \|x_1, \ldots, x_{n-k}, a_{i_1}, \ldots, a_{i_k}\|^2 \right)^{1/2}, \quad (2)$$

defines an $(n-k)$-norm on $X$ (see [5] and [11] for similar results). In particular, the triangle inequality can be verified as follows:

$$\|x + y, x_2, \ldots, x_{n-k}\|$$

$$= \left( \sum_{\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}} \|x + y, x_2, \ldots, x_{n-k}, a_{i_1}, \ldots, a_{i_k}\|^2 \right)^{1/2}$$

$$\leq \left( \sum_{\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}} \left( \|x, x_2, \ldots, x_{n-k}, a_{i_1}, \ldots, a_{i_k}\| \right. \right.$$  

$$\left. + \|y, x_2, \ldots, x_{n-k}, a_{i_1}, \ldots, a_{i_k}\| \right)^2 \right)^{1/2}$$

$$\leq \left( \sum_{\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}} \|x, x_2, \ldots, x_{n-k}, a_{i_1}, \ldots, a_{i_k}\|^2 \right)^{1/2}$$
The first inequality follows from the triangle inequality for the $n$-norm, while the second one follows from the triangle inequality for the $l^2$-type norm.

In general, for $1 \leq p \leq \infty$, one may observe that

$$
\|x_1, \ldots, x_{n-k}\| := \left( \sum_{\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}} \|x_1, \ldots, x_{n-k}, a_{i_1}, \ldots, a_{i_k}\|^p \right)^{1/p},
$$

also defines an $(n-k)$-norm on $X$. Among these derived $(n-k)$-norms, however, the case $p = 2$ is special in the following sense.

**Fact 2.4.** If the $n$-norm satisfies the parallelogram law

$$
\|x+y, x_2, \ldots, x_n\|^2 + \|x-y, x_2, \ldots, x_n\|^2 = 2(\|x, x_2, \ldots, x_n\|^2 + \|y, x_2, \ldots, x_n\|^2),
$$

then the derived $(n-k)$-norm given by (2) satisfies

$$
\|x+y, x_2, \ldots, x_{n-k}\|^2 + \|x-y, x_2, \ldots, x_{n-k}\|^2 = 2(\|x, x_2, \ldots, x_{n-k}\|^2 + \|y, x_2, \ldots, x_{n-k}\|^2).
$$

In particular, the derived norm satisfies

$$
\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).
$$

**Proof.** There are two ways to prove it. The first one is by establishing the parallelogram law directly. Indeed, by definition and hypothesis, we have

$$
\|x+y, x_2, \ldots, x_{n-k}\|^2 + \|x-y, x_2, \ldots, x_{n-k}\|^2
$$

$$
= \sum_{\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}} \left( \|x+y, x_2, \ldots, x_{n-k}, a_{i_1}, \ldots, a_{i_k}\|^2 

+ \|x-y, x_2, \ldots, x_{n-k}, a_{i_1}, \ldots, a_{i_k}\|^2 \right)
$$

$$
= \sum_{\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}} 2 \left( \|x, x_2, \ldots, x_{n-k}, a_{i_1}, \ldots, a_{i_k}\|^2 + \|y, x_2, \ldots, x_{n-k}, a_{i_1}, \ldots, a_{i_k}\|^2 \right)
$$

$$
= 2(\|x, x_2, \ldots, x_{n-k}\|^2 + \|y, x_2, \ldots, x_{n-k}\|^2).
$$
as desired.

The second way to prove it is by defining an $n$-inner product $\langle \cdot, \cdot, \ldots, \cdot \rangle_n$ on $X$ by

$$\langle x, y| x_2, \ldots, x_n \rangle := \frac{1}{4} (\|x + y, x_2, \ldots, x_n\|^2 - \|x - y, x_2, \ldots, x_n\|^2),$$

and deriving an $(n - k)$-inner product from it with respect to $\{a_1, \ldots, a_n\}$. One will then realize that the derived $(n - k)$-norm is the induced $(n - k)$-norm from the derived $(n - k)$-inner product, and hence the parallelogram law follows.

### 2.2. Finite-dimensional case

Suppose here that $(X, \langle \cdot, \cdot, \ldots, \cdot \rangle_n)$ is an $n$-inner product space of finite-dimension $d \geq n$. Then one can derive an $(n - k)$-inner product from the $n$-inner product in a slightly different way. To be precise, take a linearly independent set $\{a_1, \ldots, a_m\}$ in $X$, with $n \leq m \leq d$. With respect to $\{a_1, \ldots, a_m\}$, define for each $k \in \{1, \ldots, n - 1\}$ the function $\langle \cdot, \cdot, \ldots, \cdot \rangle_{n-k}$ on $X^{n-k+1}$ by

$$\langle x, y| x_2, \ldots, x_{n-k} \rangle := \sum_{\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, m\}} \langle x, y| x_2, \ldots, x_{n-k}, a_{i_1}, \ldots, a_{i_k} \rangle.$$ 

Then we have:

**Fact 2.5.** The function $\langle \cdot, \cdot, \ldots, \cdot \rangle_{n-k}$ defines an $(n-k)$-inner product on $X$.

**Proof.** Similar to the proof of Fact 2.1.

As we shall see in the next section, we may obtain an interesting inner product from the $n$-inner product by using a set of $d$, rather than just $n$, linearly independent vectors in $X$ (that is, by using a basis for $X$).

### 3. Examples

We shall here present some examples showing us what sort of inner products can be derived through (1) when the $n$-inner product is simple, and how they are related to the original inner product.
Example 3.1. Let $X = \mathbb{R}^n$ be equipped with the standard $n$-inner product

$$\langle x, y \rangle_{x_2, \ldots, x_n} := \begin{vmatrix} x \cdot y & x \cdot x_2 & \cdots & x \cdot x_n \\ x_2 \cdot y & x_2 \cdot x_2 & \cdots & x_2 \cdot x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_n \cdot y & x_n \cdot x_2 & \cdots & x_n \cdot x_n \end{vmatrix},$$  \hspace{1cm} (3)

where $x \cdot y$ is the usual inner product on $\mathbb{R}^n$. Then one may observe that the derived $(n-k)$-inner product with respect to an orthonormal basis $\{b_1, \ldots, b_n\}$ coincides with the standard $(n-k)$-inner product on $\mathbb{R}^n$, that is,

$$\langle x, y \rangle_{x_2, \ldots, x_{n-k}} := \begin{vmatrix} x \cdot y & x \cdot x_2 & \cdots & x \cdot x_{n-k} \\ x_2 \cdot y & x_2 \cdot x_2 & \cdots & x_2 \cdot x_{n-k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n-k} \cdot y & x_{n-k} \cdot x_2 & \cdots & x_{n-k} \cdot x_{n-k} \end{vmatrix}.$$

In particular, the derived inner product $\langle x, y \rangle$ with respect to $\{b_1, \ldots, b_n\}$, which is given by

$$\langle x, y \rangle = \langle x, y \rangle_{b_2, b_3, \ldots, b_n} + \langle x, y \rangle_{b_1, b_3, \ldots, b_n} + \cdots + \langle x, y \rangle_{b_1, b_2, \ldots, b_{n-1}},$$  \hspace{1cm} (4)

is precisely the usual inner product $x \cdot y$.

This example tells us that, on $\mathbb{R}^n$, we can define the standard $n$-inner product by using the usual inner product as in (3) and, conversely, derive the usual inner product from the standard $n$-inner product via (4).

Example 3.2. Let $X = \mathbb{R}^n$ be equipped with the standard $n$-inner product as in the preceding example. Then one may verify that the derived inner product with respect to an arbitrary linearly independent set $\{a_1, \ldots, a_n\}$ in $X$ is given by

$$\langle x, y \rangle = \|a_1, \ldots, a_n\|^2 (A^{-1}x) \cdot (A^{-1}y),$$

where $A$ is the $n \times n$ matrix whose $i$-th column is the vector $a_i$. Note that, for every $i, j \in \{1, \ldots, n\}$, we have

$$\langle a_i, a_j \rangle = \|a_1, \ldots, a_n\|^2 b_i \cdot b_j,$$

where $\{b_1, \ldots, b_n\}$ is the standard basis for $\mathbb{R}^n$. This means that $\{a_1, \ldots, a_n\}$ is an orthogonal basis for $(X, \langle \cdot, \cdot \rangle)$, as remarked previously in §2.
Remark. By invoking Parseval’s identity (see, e.g., [1], p. 354), Examples 3.1 and 3.2 extend to any \( n \)-dimensional inner product space \( X \).

Example 3.3. Suppose that \( X \) is an inner product space of dimension \( d \geq n \) and \( \{e_1, \ldots, e_n\} \) is an orthonormal set in \( X \). Equip \( X \) with the standard \( n \)-inner product as in (3), with \( x \cdot y \) being the inner product on \( X \). Then one may observe that the derived inner product with respect to \( \{e_1, \ldots, e_n\} \) is given by

\[
\langle x, y \rangle = Px \cdot Py + n(Qx \cdot Qy),
\]

where \( P \) denotes the orthogonal projection on the subspace spanned by \( \{e_1, \ldots, e_n\} \) and \( Q = I - P \) is its complementary projection. Notice here that its induced norm is equivalent to the original norm.

Although a little bit messy, it is also possible to obtain the expression for the derived \((n-1)\)-inner product for each \( k \in \{1, \ldots, n-1\} \). For example, the derived \((n-1)\)-inner product with respect to \( \{e_1, \ldots, e_n\} \) is given by

\[
\langle x, y| x_2, \ldots, x_{n-1} \rangle
\]

with \( x_1 \) being identified as \( x \).

Example 3.4. Let \( X = \mathbb{R}^d \) be equipped with the standard \( n \)-inner product as in (3), with \( x \cdot y \) being the usual inner product on \( \mathbb{R}^d \). Then one may particularly observe that the derived inner product with respect to an orthonormal basis \( \{b_1, \ldots, b_d\} \) is given by

\[
\langle x, y \rangle = \sum_{\{i_2, \ldots, i_n\} \subseteq \{1, \ldots, d\}} \langle x, y| b_{i_2}, \ldots, b_{i_n} \rangle = C_{n-1}^{d-1} x \cdot y,
\]

where \( C_{n-1}^{d-1} = \frac{(d-1)!}{(d-n)(n-1)!} \). This derived inner product is better than the previous one in the sense that it is only a multiple of the usual inner product.
Remark. By invoking Parseval’s identity, Example 3.4 may also be extended to any finite $d$-dimensional inner product space $X$.

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